International Journal of Control

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Shuyou Yu [a] , Christoph Böhm [a] , Hong Chen [b] & Frank Allgöwer [a]

[a] Institute for Systems Theory and Automatic Control, University of Stuttgart, Stuttgart, Germany

[b] Department of Control Science and Engineering, Jilin University, Changchun, P.R. China
Version of record first published: 21 Feb 2012.

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Shuyou Yu[a]*, Christoph Böhm[a], Hong Chen[b] and Frank Allgöwer[a]

[a]*Institute for Systems Theory and Automatic Control, University of Stuttgart, Stuttgart, Germany;*
[b]*Department of Control Science and Engineering, Jilin University, Changchun, P.R. China*

This article considers robust model predictive control (MPC) schemes for linear parameter varying (LPV) systems in which the time-varying parameter is assumed to be measured online and exploited for feedback. A closed-loop MPC with a parameter-dependent control law is proposed first. The parameter-dependent control law reduces conservativeness of the existing results with a static control law at the cost of higher computational burden. Furthermore, an MPC scheme with prediction horizon '1' is proposed to deal with the case of asymmetric constraints. Both approaches guarantee recursive feasibility and closed-loop stability if the considered optimisation problem is feasible at the initial time instant.

Keywords: model predictive control; linear parameter varying systems; prediction horizon '1'; convex optimisation problem

1. Introduction

Model predictive control (MPC) or receding horizon control is a class of optimisation-based control methods in which a control sequence is determined by optimising a finite horizon cost at each sampling instant, based on an explicit process model and state measurements. The first control action of the optimal sequence is applied to the plant. At the next sampling instant, the optimisation problem is solved again with new measurements, and the control input is updated. Due to its ability to handle constraints on inputs and states, the method has received much interest in both academic community and industrial society over the last 30 years (see, e.g. Mayne, Rawlings, Rao, and Scokaert 2000; Qin and Badgwell 2003).

Linear parameter varying (LPV) systems are linear systems whose dynamics depend on time-varying parameters, which take their values in pre-specified sets. Usually, it is assumed that the parameters can be measured. The analysis and synthesis of LPV systems play an important role in control theory and application since both nonlinear systems and linear systems with model uncertainties can be dealt within the framework of LPV systems (Lim 1999; Scherer 2001). Predictive control of linear uncertain systems has been proposed based on the concept of ellipsoidal invariant sets (Kothare, Balakrishnan, and Morari 1996). A state-feedback control law is designed online which minimises an upper bound on the 'worst-case' infinite horizon

objective function, while at the same time keeping the system state inside an invariant feasible set. The approach (Kothare et al. 1996) is a suitable choice as MPC controllers for LPV systems since they robustly stabilise an LPV system for all possible parameter variations. However, it has not explicitly been developed for LPV systems and therefore suffer from rather conservative linear matrix inequality (LMI) conditions that have to be satisfied. Many results in the literature represent extensions of Kothare et al. (1996), for example, schemes with enlarged feasible region and reduced computational burden have been developed. Using parameter-dependent Lyapunov functions, (Cuzzola, Geromel, and Morari 2002; Lee and Won 2006; Wada, Saito, and Saeki 2006) propose procedures which do not require the quadratic stabilisability of the given system. An improved approach is proposed in Kouvaritakis, Rossiter, and Schuurmans 2000 which deploys a fixed state-feedback law with perturbations. The algorithm requires a modest amount of online computation and introduces extra degree of freedom to enlarge the volume of the relevant invariant set. The controllers suggested in Lu and Arkun (2000) and Park and Jeong (2004) are restricted to LPV systems with bounded rates of parameter variation. Those approaches are not applicable to the case considered in this article where we assume that the parameters may vary arbitrarily within a given set. The approach presented in Lu and Arkun (2000) assumes the

*Corresponding author. Email: shuyou.yu@ist.uni-stuttgart.de

parameter to be measurable in real-time. This knowledge on the parameter allows to obtain in the first step an exact prediction of the future system behaviour and therefore reduced conservatism. In the MPC controllers proposed in Lee and Won (2006) and Lee, Park, Ji, and Won (2007), the control law is independent of the system parameter. Similar to Kothare et al. (1996), those approaches robustly stabilise the considered LPV system. Thus, if the parameter is measurable, this knowledge cannot be exploited. We will show in this article that the incorporation of the parameter measurement in the control law may reduce conservatism and improve the controller performance. A solution involving the parameter measurement in the controller design is suggested in Casavola, Famulare, and Franzé (2002). However, this approach relies on conservative LMI conditions. As will be shown, those conditions can be relaxed using the results presented in Gao (2006) and Yu, Chen, Böhm, and Allgöwer (2009). Robust receding horizon control schemes are proposed in Park, Lee, and Kwon (1999) and Ding, Xi, and Li (2004) which are based on the minimisation of the worst-case stage cost with a finite terminal penalty function. However, it should be noted that only symmetric box constraints are considered in the aforementioned works which restrict the potential application of the proposed schemes.

In this article, we discuss MPC schemes for LPV systems with output constraints. First, we propose a closed-loop MPC scheme for LPV systems with a parameter-dependent control law. Compared with the result proposed in Kothare et al. (1996), the parameter-dependent control law reduces the conservativeness of the optimisation problem. Then, we propose an MPC scheme with prediction horizon '1' for LPV systems with possibly asymmetric constraints. If the system has symmetric box constraints, the optimisation problem of MPC with prediction horizon '1' can be formulated as a semi-definite programme involving LMIs. Recursive feasibility of the optimisation problems and stability of the closed-loop system are guaranteed for both the proposed schemes. We highlight that the LMIs developed in the closed-loop MPC with a parameter-dependent control law can also be used to solve the terminal control law of MPC with prediction horizon '1' since the fictitious control law is not applied to the considered systems at all.

This article is organised as follows. Section 2 introduces the class of LPV systems and presents the considered setup. Section 3 discusses a closed-loop MPC of LPV systems with symmetric box constraints. Section 4 proposes MPC with prediction horizon '1' for LPV systems with possibly asymmetric constraints. Numerical examples to illustrate the effectiveness of the proposed algorithm are given in Section 5.

Notation: The following notation is used throughout this article. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the field of real number, the set of integer numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The notations $\mathbb{Z}_{[c_1,c_2]}$ and $\mathbb{Z}_{[c_1,c_2)}$ denote the sets $\{k \in \mathbb{Z} \mid c_1 \leq k \leq c_2\}$ and $\{k \in \mathbb{Z} \mid c_1 \leq k < c_2\}$. Suppose that $M \in \mathbb{R}^{n \times n}$, M^T denotes the transpose of M and $\bar{\sigma}(M)$ denotes the largest singular value of matrix M. I_m denotes the $m \times m$ identity matrix, and $*$ denotes the corresponding symmetric block in symmetric matrices.

2. LPV systems

Consider discrete-time LPV systems of the form

$$x_{k+1} = A(\theta_k)x_k + B(\theta_k)u_k, \tag{1a}$$

$$y_k = C(\theta_k)x_k + D(\theta_k)u_k, \tag{1b}$$

subject to the constraints

$$y_k \in \mathcal{H}, \tag{2}$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the state, $u_k \in \mathbb{R}^{n_u}$ the control input and $y_k \in \mathbb{R}^{n_y}$ the performance output. The output y_k cannot necessarily be measured. The compact set $\mathcal{H}$ contains the origin in its interior. If the constraints are in a symmetric box, we can write them in an element-wise fashion as

$$-y_{m,\max} \leq y_{m,k} \leq y_{m,\max}, \quad m \in \mathbb{Z}_{[1,n_y]}, \tag{3}$$

where $y_{\max} := [y_{1,\max}, \ldots, y_{n_y,\max}]^T$ is a given constant vector of compatible size.

The system matrices $A(\theta_k) \in \mathbb{R}^{n_x \times n_x}$, $B(\theta_k) \in \mathbb{R}^{n_x \times n_u}$, $C(\theta_k) \in \mathbb{R}^{n_y \times n_x}$ and $D(\theta_k) \in \mathbb{R}^{n_y \times n_u}$ are assumed to depend linearly on the parameter vector $\theta_k := [\theta_{1,k}, \theta_{2,k}, \ldots, \theta_{N,k}]^T \in \mathbb{R}^N$, which belongs to a convex polytope $\mathcal{P}$ defined by

$$\mathcal{P} := \left\{ \theta_k \in \mathbb{R}^N \,\middle|\, \sum_{j=1}^{N} \theta_{j,k} = 1, \ \bar{\theta}_{j,k} \geq 0 \right\}. \tag{4}$$

N is the number of matrix vertices. Clearly, as θ_k varies inside the convex polytope $\mathcal{P}$, the matrices of the system (1) vary inside a corresponding polytope Ψ, which is defined by the convex hull of N local matrix vertices $[A_i, B_i, C_i, D_i]$, $i \in \mathbb{Z}_{[1,N]}$,

$$\Psi := \mathrm{Co}\left\{ \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \ldots, \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \right\}.$$

Therefore, we can write the matrices of the system (1) as

$$A(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} A_j, \quad B(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} B_j,$$
$$C(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} C_j, \quad D(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} D_j. \tag{5}$$

2.1 Control task and problem setup

The control task is to stabilise the origin of the system (1) with an MPC law such that the constraints (2) are satisfied. To distinguish actual state and predicted trajectories, in what follows the index $k + i/k$ denotes future values at time $k + i$ predicted at time k, $i \in \mathbb{Z}_{[0,\infty)}$.

In this article, we first consider an infinite horizon optimisation problem.

Problem 1: Let $Q \in \mathbb{R}^{n_x \times n_x}$ and $R \in \mathbb{R}^{n_u \times n_u}$ be positive definite weighting matrices. Consider the system (1)–(2). For a given initial state x_k, find a control law $u_{k+i/k} = \kappa(x_{k+i/k})$ such that the infinite horizon cost functional

$$J_{\infty/k}(x_k) = \max_{\theta_{k+i/k} \in \mathcal{P}} \sum_{i=0}^{\infty} \left\{ x_{k+i/k}^T Q x_{k+i/k} + u_{k+i/k}^T R u_{k+i/k} \right\} \tag{6}$$

is minimised with respect to the time-varying parameter $\theta_{k+i/k}$ and the output constraints (2) at each sampling instant k, based on a prediction of the system behaviour into the future, and $x_{k/k} := x_k$.

Assume that no mismatch exists between the model and the plant, and that both the parameter vector θ_k and the state x_k are measured in real-time. Thus, at every sampling time k the current system matrices

$$\begin{bmatrix} A(\theta_{k/k}) & B(\theta_{k/k}) \\ C(\theta_{k/k}) & D(\theta_{k/k}) \end{bmatrix}$$

are known exactly but the future system matrices

$$\begin{bmatrix} A(\theta_{k+i/k}) & B(\theta_{k+i/k}) \\ C(\theta_{k+i/k}) & D(\theta_{k+i/k}) \end{bmatrix}$$

are uncertain, since we cannot know exactly the future behaviour of the system parameter $\theta_{k+i/k}$, $i \in \mathbb{Z}_{[1,\infty)}$. Notice that they vary inside the polytope Ψ. Therefore, in the cost functional (6) the worst case over all possible future parameters has to be considered.

3. Closed-loop MPC with a parameter-dependent control law for LPV systems

In this section, we propose a new MPC law for the system (1) subject to the symmetric box constraints (3)

by using a parameter-dependent state feedback control law, which is obtained via the solution of a convex optimisation problem. The obtained LMI conditions provide more degrees of freedom in the controller design compared with the scheme which has a static feedback control law.

Consider a linear parameter-dependent state feedback control law

$$u_k = K(\theta_k) x_k, \tag{7}$$

which is updated at each sampling instant via the minimisation of an upper bound on the cost functional (6). Suppose that $K_j \in \mathbb{R}^{n_u \times n_x}$ is a time-invariant feedback gain associated with the j-th vertex system. A suitable parameter-dependent feedback law for the whole LPV system is obtained via the weighted average of the control laws designed for each vertex

$$K(\theta_k) = \sum_{j=1}^{N} \theta_{j,k} K_j. \tag{8}$$

For the system (1), using the controller (7)–(8), we obtain the closed-loop representation

$$x_{k+1} = A_{cl}(\theta_k) x_k$$
$$z_k = C_{cl}(\theta_k) x_k, \tag{9}$$

where

$$A_{cl}(\theta_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k} \theta_{j,k} (A_i + B_i K_j) \quad \text{and}$$

$$C_{cl}(\theta_k) = \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k} \theta_{j,k} (C_i + D_i K_j).$$

3.1 Upper bound on the cost functional

The solution to the following optimisation problem provides an upper bound of the cost functional (6) at time instant k, as will be shown in Theorem 1.

Problem 2: At time k, measure the state x_k and the parameter vector θ_k, consider the optimisation problem

$$\underset{\gamma_k, X_k, Y_{1,k}, Y_{2,k}, \dots, Y_{N,k}}{\text{minimise}} \gamma_k \tag{10a}$$

subject to

$$\begin{bmatrix} 1 & x_k^T \\ x_k & X_k \end{bmatrix} \geq 0, \tag{10b}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k+v/k} \theta_{j,k+v/k} L_{ij} \geq 0, \quad v \in \mathbb{Z}_{[0,\infty)}, \tag{10c}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{i,k+v/k} \theta_{j,k+v/k} F_{ij,m} \geq 0, \quad m \in \mathbb{Z}_{[1,n_y]}, \tag{10d}$$

with the matrices

$$
L_{ij} = \begin{bmatrix} X_k & * & * & * \\ A_i X_k + B_i Y_{j,k} & X_k & * & * \\ Q^{\frac{1}{2}} X_k & 0 & \gamma_k I & * \\ R^{\frac{1}{2}} Y_{j,k} & 0 & 0 & \gamma_k I \end{bmatrix},
$$

$$
F_{ij,m} = \begin{bmatrix} y_{m,\max}^2 & e_m^T (C_i X_k + D_i Y_{j,k}) \\ * & X_k \end{bmatrix},
$$

and $e_m := [0, \ldots, 0, 1, 0, \ldots, 0]^T \in \mathbb{R}^{n_y}$ with its m-th element '1' and the other elements '0'.

As discussed in Section 2, the parameter vector $\theta_{k/k}$ is known, and the future vector $\theta_{k+i/k}$, $i \in \mathbb{Z}_{[1,\infty)}$, varies inside a convex polytope $\mathcal{P}$.

Remark 3.1: Note that for simplicity of notation we have skipped the index k in the matrices L_{ij} and $F_{ij,m}$. It is clear from the definition of those matrices that they change with k since they depend on X_k and $Y_{j,k}$.

The following theorem derives conditions to obtain an upper bound on the cost functional (6) using the system description (9) and Problem 2.

Theorem 1: *Suppose that there exist a symmetric, positive definite matrix $X_k \in \mathbb{R}^{n_x \times n_x}$, matrices $Y_{j,k} \in \mathbb{R}^{n_u \times n_x}$, $j \in \mathbb{Z}_{[0,N]}$, and a constant $\gamma_k > 0$ such that Problem 2 at time instant k has a feasible solution for all $\theta_{k+v/k} \in \mathcal{P}$, $v \in \mathbb{Z}_{[1,\infty)}$, where x_k is the measured system state at the sampling instant k. Then, with $P_k = \gamma_k X_k^{-1}$, $K_{j,k} = Y_{j,k} X_k^{-1}$, $j \in \mathbb{Z}_{[1,N]}$, and the parameter-dependent control law*

$$
u_{k+v/k} = K(\theta_{k+v/k}) x_{k+v/k}, \tag{11}
$$

where $K(\theta_{k+v/k}) = \sum_{j=1}^{N} \theta_{j,k+v/k} K_{j,k}$, the following holds:

(1) *The predicted states $x_{k+v/k}$ with $x_{k/k} = x_k$ converge to the origin as $v \to \infty$.*
(2) *The expression $V_k = x_k^T P_k x_k$ is minimised and represents an upper bound on the cost functional (6) at the sampling instant k.*
(3) *The constraints (3) are satisfied for all $v \in \mathbb{Z}_{[0,\infty)}$.*

Remark 3.2: The solution to Problem 2 is a feasible solution to Problem 1, and the term γ_k of Equation (10a) is an upper bound on the cost functional (6).

3.2 *Optimisation problem and properties*

Theorem 1 gives conditions for the minimisation of an upper bound on the infinite horizon cost functional (6). However, the matrix inequalities (10c) and (10d) depend on the unknown future parameter $\theta_{k+v/k}$ for all $v \in \mathbb{Z}_{[1,\infty)}$. This makes it impossible to find a

solution to Problem 2. To deal with the difficulty, we introduce the following lemma.

Lemma 1 (Kim and Lee 2000): *If there exist matrices $\Lambda_{ij} = \Lambda_{ji}^T$, $i, j \in \mathbb{Z}_{[1,N]}$, such that the LMIs*

$$
\Gamma_{ii} \geq \Lambda_{ii}, \tag{12a}
$$

$$
\Gamma_{ij} + \Gamma_{ji} \geq \Lambda_{ij} + \Lambda_{ij}^T, \tag{12b}
$$

$$
[\Lambda_{ij}]_{N \times N} \geq 0, \quad j < i, \tag{12c}
$$

are satisfied, where

$$
[\Lambda_{ij}]_{N \times N} = \begin{bmatrix} \Lambda_{11} & \cdots & \Lambda_{1N} \\ \vdots & \ddots & \vdots \\ \Lambda_{N1} & \cdots & \Lambda_{NN} \end{bmatrix}, \tag{13}
$$

then, with $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$, the parameter-dependent matrix inequalities

$$
\sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j \Gamma_{ij} \geq 0 \tag{14}
$$

are satisfied.

We observe that Lemma 1 reformulates the parameter-dependent matrix inequality (14) as a set of LMI conditions (12). In virtue of this result, we introduce an infinite horizon MPC scheme for LPV systems subject to constraints by applying Lemma 1 to Equations (10c)–(10d).

We consider the following optimisation problem which can be treated as a convex optimisation problem with respect to LMI constraints.

Problem 3: At time k, measure the state x_k and the parameter vector θ_k, and solve the optimisation problem

$$
\underset{\gamma_k, X_k, Y_{1,k}, Y_{2,k}, \ldots, Y_{N,k}, T_{ij}, S_{ij}}{\text{minimise}} \gamma_k \tag{15a}
$$

subject to

$$
\begin{bmatrix} 1 & x_k^T \\ x_k & X_k \end{bmatrix} \geq 0, \tag{15b}
$$

$$
L_{ii} \geq T_{ii}, \tag{15c}
$$

$$
L_{ij} + L_{ji} \geq T_{ij} + T_{ij}^T, \tag{15d}
$$

$$
[T_{ij}]_{N \times N} \geq 0, \quad i \in \mathbb{Z}_{[1,N]}, \, j < i, \tag{15e}
$$

$$
F_{ii,m} \geq S_{ii,m}, \tag{15f}
$$

$$F_{ij,m} + F_{ji,m} \geq S_{ij,m} + S_{ij,m}^T, \qquad (15g)$$

$$\left[S_{ij,m} \right]_{N \times N} \geq 0, \quad i \in \mathbb{Z}_{[1,N]}, \; j < i \qquad (15h)$$

where L_{ij} and $F_{ij,m}$ are as defined in Problem 2.

The MPC control law is defined by the following algorithm.

Algorithm 1:

Step 1: At time k, measure the state x_k and the parameter vector θ_k, and solve Problem 3.

Step 2: Apply the control input

$$u_k := K(\theta_{k|k}) x_k \qquad (16)$$

to the system (1), where $K(\theta_{k|k})$ is the first part of the optimal feedback sequence $K(\theta_{k+v|k}) = \sum_{j=1}^{N} \theta_{j,k+v|k} K_{j,k}$, $K_{j,k} := Y_{j,k} X_k^{-1}$, for all $j \in \mathbb{Z}_{[1,N]}$ and $v \in \mathbb{Z}_{[0,\infty)}$.

Step 3: Set $k := k+1$ and go to Step 1.

Theorem 2: *Consider the LPV system* (1) *subject to the constraints* (3) *and the cost functional* (6)*. Problem 3, which is solved repeatedly at each sampling instant k, has the following properties*:

(1) *Problem 3 is convex. Furthermore, it is feasible at all future sampling instant if it is feasible at the initial time instant.*
(2) *The solution to Problem 3 minimises the upper bound $V_k = x_k^T P_k x_k$ on cost functional* (6) *at each sampling instant k, with $P_k = \gamma_k X_k^{-1}$.*
(3) *If Problem 3 is initially feasible, the control law* (16) *asymptotically stabilises the origin of system* (1)*.*
(4) *The MPC control law* (16) *ensures that the symmetric box constraints* (3) *are satisfied for all k.*

Remark 3.3: Algorithm 1 together with Problem 3 provides a closed-loop MPC scheme on Problem 1. Similar to the scheme proposed in Kothare et al. (1996), a feedback control law is adopted at each time instant. However, the control law is parameter-dependent rather than static.

Remark 3.4: It is worth pointing out that the proposed MPC control law is less conservative than those suggested in Kothare et al. (1996) and Casavola et al. (2002). For example, one can see that the solution to the optimisation problem in Kothare et al. (1996) and Casavola et al. (2002) should satisfy the condition $L_{i1} > 0$ for all $i \in \mathbb{Z}_{[1,N]}$. Thus, Problem 3 has a larger set of feasible solutions compared with the optimisation problems in either Casavola et al. (2002) or Kothare et al. (1996).

In the following section, we propose an MPC scheme for LPV systems subject to possibly asymmetric constraints, which adopts the analogous framework of a terminal control law, a terminal set and a terminal penalty of quasi-infinite horizon MPC by using already existing information.

4. MPC with prediction horizon '1' for LPV systems

In this section, we propose an MPC scheme with prediction horizon '1' for LPV systems which has possibly asymmetric constraints. Stability of the closed-loop system and recursive feasibility of the related optimisation problem can be guaranteed if the optimisation problem is feasible at the initial time instant. Furthermore, if the constraints are in a symmetric box, the optimisation problem can be formulated as a convex optimisation problem formulated via LMIs.

4.1 MPC of LPV systems with possibly asymmetric constraints

We consider general constraints (2) in this subsection, i.e. not necessarily symmetric box constraints. Given $\alpha > 0$, define $\mathcal{X}_f$ as a neighbourhood of the origin

$$\mathcal{X}_f := \{ x \in \mathbb{R}^{n_x} | E(x) \leq \alpha \}. \qquad (17)$$

$\mathcal{X}_f$ is a level set of the positive definite function $E(x) := x^T P x$, where $P \in \mathbb{R}^{n_x \times n_x}$ is a positive definite matrix. Similar to quasi-infinite horizon nonlinear MPC (Chen and Allgöwer 1998), the following definition is needed to reformulate the infinite horizon cost functional (6).

Definition 1: $\mathcal{X}_f$ and $E(x)$ are said to be the terminal set and the terminal penalty function, respectively, suppose that there exists a continuous local control law $u = \pi(x)$ such that

B0 $\mathcal{X}_f \in \mathcal{X}$, where $\mathcal{X}$ is the projection of output space $\mathcal{H}$ to the state space.
B1 $y_k \in \mathcal{H}$ for all $x_k \in \mathcal{X}_f$,
B2 $E(x)$ satisfies the inequality

$$E(x_{k+i+1}) - E(x_{k+i}) + x_{k+i}^T Q x_{k+i}$$
$$+ \pi(x_{k+i})^T R \pi(x_{k+i}) \leq 0, \qquad (18)$$

for all $\theta_{k+i} \in \mathcal{P}$ and all $x_{k+i} \in \mathcal{X}_f$, $i \in \mathbb{Z}_{[0,\infty)}$.

According to Definition 1, $\mathcal{X}_f$ has the following properties:

(1) The origin is contained in the interior of $\mathcal{X}_f$, since $E(x) > 0$ for all $x \in \mathcal{X}_f \backslash \{0\}$,
(2) $\mathcal{X}_f$ is closed and connected since $E(x)$ is continuous in x,

(3) Since (18) holds, $\mathcal{X}_f$ is a robust positive invariant set for the 'unknown' parameter of the LPV system (1) controlled by $u = \pi(x)$.

Remark 4.1: The terminal set in Chen and Allgöwer (1998) is a neighbourhood of the equilibrium which satisfies the constraints and is positive invariant. Here, $\mathcal{X}_f$ is 'robust' positive invariant for all admissible but 'unknown' parameters.

Lemma 2: *Following Definition* 1, *the terminal set* $\mathcal{X}_f$ *and the terminal cost function* $E(x)$ *are such that*

$$E(x_{k+1/k})$$
$$\geq \max_{\theta_{k+i/k} \in \mathcal{P}} \sum_{i=1}^{\infty} \left\{ x_{k+i/k}^T Q x_{k+i/k} + \pi(x_{k+i/k})^T R \pi(x_{k+i/k}) \right\}$$

for all $x \in \mathcal{X}_f$.

Using Lemma 2, we approximate the infinite horizon cost functional as a one-step ahead functional

$$J_k(x_{k/k}) := x_{k/k}^T Q x_{k/k} + u_{k/k}^T R u_{k/k} + x_{k+1/k}^T P x_{k+1/k}, \tag{19}$$

which is an upper bound of the cost functional $J_{\infty/k}(x_{k/k})$. Therefore, Problem 1 for the current state x_k and the current parameter θ_k can be reformulated as follows.

Problem 4: At time k, measure the state x_k and the parameter vector θ_k, and solve the optimisation problem

$$\underset{u_{k/k}}{\text{minimise}} \quad J_k(x_{k/k})$$

subject to

$$x_{k+1/k} = A(\theta_k)x_{k/k} + B(\theta_k)u_{k/k}, \quad x_{k/k} = x_k,$$

$$y_{k/k} = C(\theta_k)x_{k/k} + D(\theta_k)u_{k/k},$$

$$y_{k/k} \in \mathcal{H},$$

$$x_{k+1/k} \in \mathcal{X}_f,$$

where $\mathcal{X}_f$ is the terminal set, $x_{k+1/k}^T P x_{k+1/k}$ is the terminal penalty function, and $J_k(x_{k/k})$ as in (19).

The MPC control law is defined as follows.

Algorithm 2

Step 1: At the sampling time $k \in \mathbb{Z}_{[0,\infty)}$, measure the state x_k and the parameter vector θ_k, and solve Problem 4.

Step 2: Apply the control input

$$u_k := \arg\min_{u_{k/k}} J_k(x_{k/k}) \tag{20}$$

to the system (1).

Step 3: Set $k = k + 1$ and go to Step 1.

A free control action $u_{k/k}$ calculated online and a fictitious control law $\pi(\cdot)$ calculated offline to determine P are needed in Problem 4, where $u_{k/k}$ is not necessarily equal to $\pi(x_k)$. However, only the free control action $u_{k/k}$ is actually applied to the system.

Remark 4.2: Comparing Problem 4 with Problem 3, where a control law $K(\theta_k)$ is calculated online and applied to the system, the free control action $u_{k/k}$ provides a degree of freedom which allows to deal with asymmetric constraints, and possibly leads to better performance.

Remark 4.3: If $\mathcal{H}$ is a convex set, then Problem 4 is a convex optimisation problem (Boyd and Vandenberghe 2004).

Remark 4.4: If not only the parameter but also its rate of variation are available online, the prediction horizon in (19) can be chosen as $N = 2$, i.e.

$$J_k(x_{k/k}) := \sum_{i=0}^{1} \left\{ x_{k+i/k}^T Q x_{k+i/k} + u_{k+i/k}^T R u_{k+i/k} \right\}$$
$$+ x_{k+2/k}^T P x_{k+2/k},$$

since $x_{k+2/k}$ can also be calculated exactly at time k.

Remark 4.5: If the varying parameter $\theta(\cdot)$ is a function of the state $x(\cdot)$ and the control $u(\cdot)$, we can choose the prediction horizon N arbitrarily large. In this case, $\theta_{k+i/k}$ can be determined by $x_{k+j/k}$ and $u_{k+j/k}$, for all $j \in \mathbb{Z}_{[0,i]}$.

According to the principle of moving horizon strategy, Problem 4 can be solved repeatedly at each time instant k based on the measurements x_k and θ_k. The following theorem investigates the properties of the system (1) under the proposed MPC law.

Theorem 3: *Suppose that*

(a) *for the LPV system* (1), *there exists a locally asymptotically stabilising controller* $u = \pi(x)$, *a continuously differentiable, positive definite function* $E(x) = x^T P x$ *that locally satisfies* (18) *and a positive invariant set* $\mathcal{X}_f$ *defined by* (17),

(b) *Problem* 4 *is feasible at the initial time* $k = 0$.

Then,

(1) *Problem* 4 *is feasible for all* $k \in \mathbb{Z}_{[0,\infty)}$,

(2) *the system* (1) *under MPC control law* (20) *according to Algorithm* 2 *is robustly asymptotically stable within the region of attraction* $\mathcal{D}$, *where* $\mathcal{D}$ *is the set of all states for which Problem* 4 *has a feasible solution.*

4.2 *Terminal set and terminal penalty function*

In order to specify Problem 4, the terminal set $\mathcal{X}_f$ as well as the terminal penalty function $x^T P x$ which is an upper bound on the infinite horizon cost functional has to be chosen *a priori*. In what follows, we state LMI conditions which can be used to determine the terminal set associated with a time-invariant or a parameter-varying terminal control law, while the constraints are in a *symmetric box*.

Remark 4.6: For LPV systems with asymmetric constraints, to the author's best knowledge, there is no systematic way to get an ellipsoidal terminal set except for choosing tighter symmetric box constraints within the asymmetric bounds.

First we present an LMI scheme to obtain the terminal set of LPV systems, which has a fixed terminal control law. The original theorem was proposed by Chen, O'Reilly, and Ballance (2003), and is about the terminal set of nonlinear systems described by linear difference inclusions.

Lemma 3 (Chen et al. 2003, Static terminal control law): *Suppose that the LPV system* (1) *is subject to symmetric box constraints* (3). *If there exist a scalar* $\alpha > 0$, *a positive definite matrix* $X \in R^{n_x \times n_x}$ *and a non-quadratic matrix* $Y \in R^{n_y \times n_x}$ *such that*

$$\begin{bmatrix} X & * & * & * \\ A_i X + B_i Y & X & * & * \\ X & 0 & \alpha Q^{-1} & * \\ Y & 0 & 0 & \alpha R^{-1} \end{bmatrix} > 0, \quad (21a)$$

$$\begin{bmatrix} y_{m,\max}^2 & e_m^T(C_i X + D_i Y) \\ * & X \end{bmatrix} \geq 0, \quad m \in \mathbb{Z}_{[1,n_y]}, \quad (21b)$$

with $i \in \mathbb{Z}_{[1,N]}$, *then the ellipsoid* $\mathcal{X}_f$ *with* $P := \alpha X^{-1}$ *and* $E(x) := x^T P x$ *can serve as a terminal set and a terminal penalty function, respectively. The associated terminal controller is* $\pi(x) := Y X^{-1} x$.

The above lemma yields a parameter-independent terminal control law. The following lemma shows that a less conservative result is obtained by using a parameter-dependent terminal control law.

Lemma 4 (Parameter-dependent terminal control law): *Suppose that the LPV system* (1) *is subject to symmetric box constraints* (3). *If there exist a scalar* $\alpha > 0$, *a positive definite matrix* $X \in R^{n_x \times n_x}$ *and matrices* $Y_j \in R^{n_y \times n_x}$, $\mathcal{T}_{ij}$ *and* $\mathcal{M}_{ij}$, $i, j \in \mathbb{Z}_{[1,N]}$ *such that* (15c)–(15h) *are satisfied, then the ellipsoid* $\mathcal{X}_f$ *with* $P := \alpha X^{-1}$ *and the function* $E(x) := x^T P x$ *serve as a terminal region and a terminal penalty function for LPV system, respectively. The associated terminal controller is* $\pi(x) := \sum_{j=1}^N \theta_j K_j x$ *with* $K_j = Y_j X^{-1}$.

Remark 4.7: In order to obtain the feasible region of Problem 4 as large as possible, one can solve the offline optimisation problem

$$\underset{\alpha, X, Y}{\text{maximise}} \ (\det X)^{\frac{1}{n_x}}, \quad \text{s.t. (21) holds,}$$

or

$$\underset{\alpha, X, Y_1, Y_2,...,Y_N}{\text{maximise}} \ (\det X)^{\frac{1}{n_x}}, \quad \text{s.t. (15c)–(15h) hold,}$$

respectively, where $\alpha > 0$ and $X > 0$, to get the fixed terminal control law or the parameter-dependent terminal control law. Both optimisation problems are convex and can be solved by standard LMI solvers (Boyd, El Ghaoui, Feron, and Balakishnan 1994).

4.3 *MPC of LPV systems with symmetric box constraints*

In this subsection, we choose the matrix P as a new *online* optimisation variable and convert Problem 4 into a convex optimisation problem with LMIs. In other words, the terminal control law, the terminal set and the terminal penalty function are determined online as well.

Minimisation of $x_{k/k}^T Q x_{k/k} + u_{k/k}^T R u_{k/k} + x_{k+1/k}^T \times P x_{k+1/k}$ with $P > 0$ is equivalent to

$$\underset{\alpha, u_{k/k}, P}{\text{minimise}} \ \alpha,$$

subject to

$$x_{k/k}^T Q x_{k/k} + u_{k/k}^T R u_{k/k} + x_{k+1/k}^T P x_{k+1/k} \leq \alpha,$$

with $\alpha > 0$. By the Schur complement, this is equivalent to

$$\underset{\alpha, u_{k/k}, X}{\text{minimise}} \ \alpha,$$

subject to

$$\begin{bmatrix} 1 & * & * & * \\ x_{k/k} & \alpha Q^{-1} & * & * \\ u_{k/k} & 0 & \alpha R^{-1} & * \\ A(\theta_k)x_{k/k} + B(\theta_k)u_{k/k} & 0 & 0 & X \end{bmatrix} \geq 0, \quad (22)$$

with $X := \alpha P^{-1}$ and $\alpha > 0$. Due to $x_{k+1/k}^T P x_{k+1/k} \leq \alpha$, which follows from (22), Problem 4 with parameter-dependent terminal control law is formulated as follows.

Problem 5: At time k, measure the state x_k and solve the optimisation problem

$$\underset{\alpha, u_{k/k}, X, Y_1, Y_2,..., Y_N}{\text{minimise}} \ \alpha,$$

subject to

$$x_{k+1/k} = A(\theta_k)x_{k/k} + B(\theta_k)u_{k/k}, \quad x_{k/k} = x_k,$$

$$y_{k/k} = C(\theta_k)x_{k/k} + D(\theta_k)u_{k/k},$$

$$- y_{m,\max} \leq y_{m,k/k} \leq y_{m,\max}, \quad \forall m \in \mathbb{Z}_{[1,n_y]}$$

$$(15c)-(15h), (22),$$

with $\alpha > 0$ and $X > 0$.

The MPC control law is defined as follows.

Algorithm 3:

Step 1: At the sampling time $k \in \mathbb{Z}_{[0,\infty)}$, measure the state x_k and solve Problem 5.

Step 2: Apply the control input

$$u_k := u_{k/k} \tag{23}$$

to the system (1), where $u_{k/k}$ is the feasible solution to Problem 5.

Step 3: Set $k = k + 1$ and go to Step 1.

Theorem 4: *Assume that Problem 5 is feasible at the initial time instant. Then, the MPC scheme according to Algorithm 3 guarantees that*

(1) *Problem 5 is feasible for all $k > 0$,*
(2) *the symmetric box constraints (3) are satisfied for all time instants,*
(3) *the MPC control law asymptotically stabilises the LPV system (1).*

Remark 4.8: If $Y = Y_1 = \cdots = Y_N$, then Problem 5 with a parameter-dependent terminal control law will degenerate into a problem with a fixed terminal control law, which has been proposed in Lu and Arkun (2000 and Park et al. (1999). In this sense, the parameter-dependent terminal control law here provides an extra degree of freedom in Problem 5, which promises a larger feasible region and a better performance, however, at the cost of heavy computational burden.

5. Numerical example

5.1 *Example 1*

We consider the two-mass-spring system (Kothare et al. 1996), as shown in Figure 1. The discrete time state space equation, which is obtained from the continuous time model using a first-order Euler
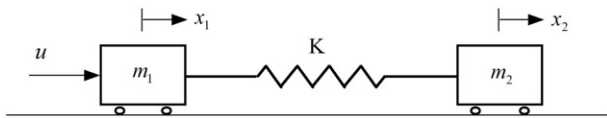

Figure 1. Coupled spring-mass system.

approximation with sample time of $\delta = 0.1$ s, is

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \\ x_{3,k+1} \\ x_{4,k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -\dfrac{0.1\mu_k}{m_1} & \dfrac{0.1\mu_k}{m_1} & 1 & 0 \\ \dfrac{0.1\mu_k}{m_2} & -\dfrac{0.1\mu_k}{m_2} & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \\ x_{4,k} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ \dfrac{0.1}{m_1} \\ 0 \end{bmatrix} u_k, \tag{24}$$

where m_1 and m_2 are the two masses and μ_k is the spring constant at the time instant k. The positions of the masses are represented by the states $x_{1,k}$ and $x_{2,k}$, whereas $x_{3,k}$ and $x_{4,k}$ describe their velocities. In this example, we choose masses $m_1 = 1$ and $m_2 = 1$, and consider the output $y_k := x_{2,k} - x_{1,k}$. The spring constant is assumed to be a time-varying function

$$\mu_k = 0.5 + 50y_k^2.$$

Due to the physical limitation of the system, we assume that an output constraint $|y(k)| \leq 0.5$ is imposed. Thus, it can be verified that $\mu_k \in [0.5, 13]$.

Introducing the parameters $\theta_{1,k} = 1 - \frac{\mu_k - 0.5}{12.5}$ and $\theta_{2,k} = 1 - \theta_{1,k}$, the system (24) can be written in the form of (5), that is, the parameters $\theta_{i,k}$, $i = 1, 2$, satisfy condition (4) and the matrices A_i and $B_i = B$, $i \in \mathbb{Z}_{[1,2]}$, are as follows:

$$A_1 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -0.05 & 0.05 & 1 & 0 \\ 0.05 & -0.05 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 0 & 0.1 & 0 \\ 0 & 1 & 0 & 0.1 \\ -1.3 & 1.3 & 1 & 0 \\ 1.3 & -1.3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0.1 \\ 0 \end{bmatrix}.$$

The control objective is to steer the example system (24) from an initial condition to the origin while satisfying the constraints $|u_k| \leq 1$ and $|y(k)| \leq 0.5$ for all k. In the example, the matrices of the infinite horizon cost functional (6) are chosen as $Q = I \in \mathbb{R}^{4 \times 4}$ and $R = 1$. The MPC with prediction horizon '1' of LPV systems with symmetric box constraints proposed in Section 4.2 is adopted. We compare our results with the approaches suggested by Lu and Arkun (2000) and Kothare et al. (1996), in order to illustrate the reduced conservativeness and the improved performance.
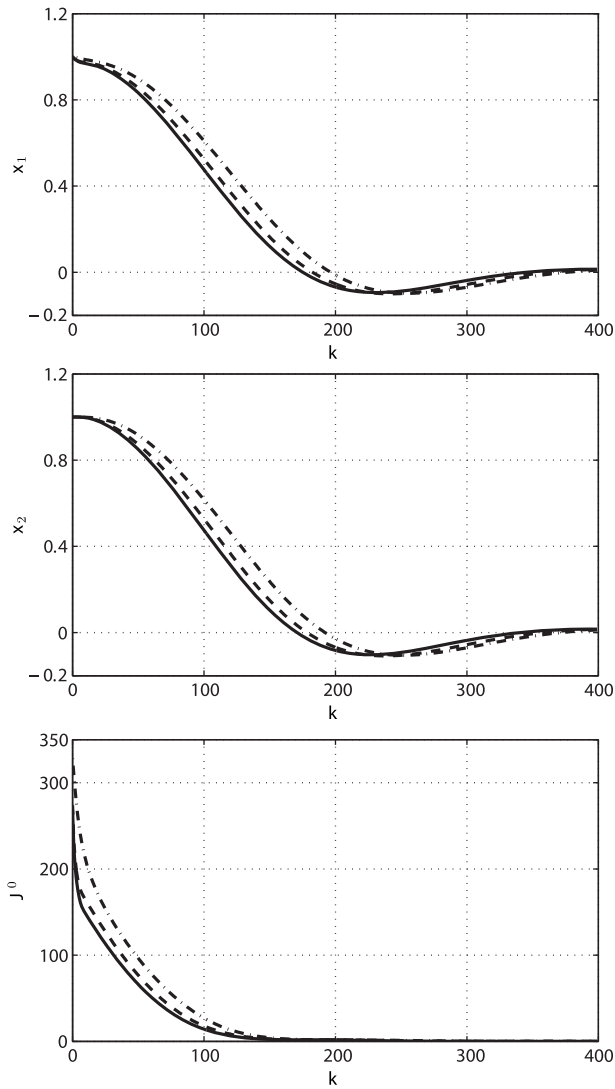
Figure 2. Comparison of the evolution of states and the performance, MPC with prediction horizon '1' for LPV system (solid line) versus the controllers of Kothare et al. (1996) (dash-dot line) and Lu and Arkun (2000) (dashed line) for the system (24) and setup, from the initial state $x_0 = [1\ 1\ 0\ 0]^T$.
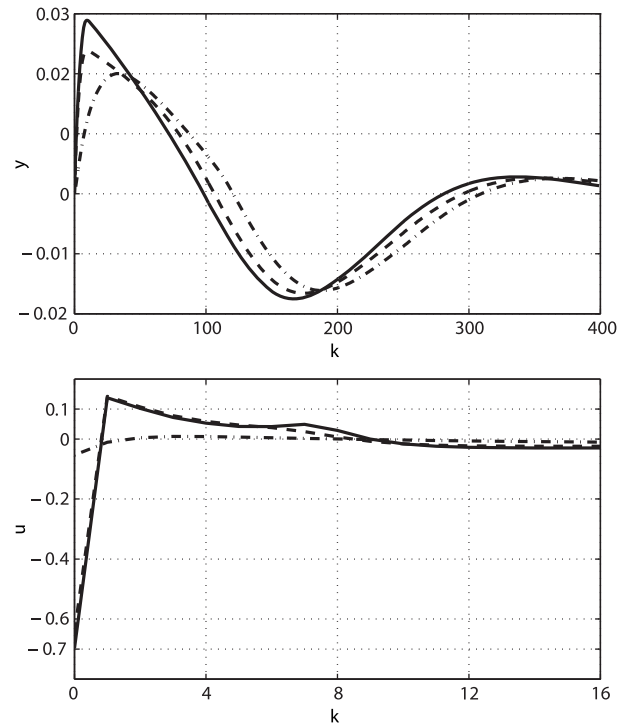

Figure 3. Comparison of both the output and the input constraints, MPC with prediction horizon '1' for LPV system (solid line) versus the controllers of Kothare et al. 1996 (dash-dot line) and Lu and Arkun 2000 (dashed line) for the system (24) and setup, from the initial state $x_0 = [1\ 1\ 0\ 0]^T$.

Figures 2 and 3 show the obtained simulation results for the closed-loop behaviour from the initial state $x_0 = [1\ 1\ 0\ 0]^T$. Compared with the approaches in Kothare et al. (1996) and Lu and Arkun (2000), the proposed MPC control law which solves Problem 5 online steers the state of the example system faster to the origin. The behaviour of the states $x_{1,k}$ and $x_{2,k}$ shows that the novel controller is able to react more efficiently on the varying parameters $\theta_{1,k}$ and $\theta_{2,k}$. This results from the parameter-dependent feedback control law and from the less conservative LMI conditions in the optimisation problem. The reduced conservativeness of the parameter-dependent control law is also

illustrated well by the behaviour of J_k^0 which represents the minimised upper bound on the worst-case cost functional. Figure 2 clearly shows that, with the parameter-dependent control law a smaller upper bound can be calculated at each sampling instant k. Figure 3 shows that both the input and the output constraints considered in this example are satisfied. Note that in identifying the differences, we only display the first few control inputs.

5.2 Example 2

Consider the time-varying discrete-time nonlinear system

$$x_{1,k+1} = 0.5x_{1,k}^2 + u_k,$$
$$x_{2,k+1} = (0.5 + \sin^2(k)/3)x_{1,k} + x_{2,k} + u_k,$$

with the asymmetric input constraint $-1.0 \leq u_k \leq 0.5$ and the state constraints $-1 \leq x_1 \leq 1$ and $-1 \leq x_2 \leq 1$. The control objective is to steer the system from the initial state to the equilibrium while satisfying the constraints. It is possible to embed the nonlinear
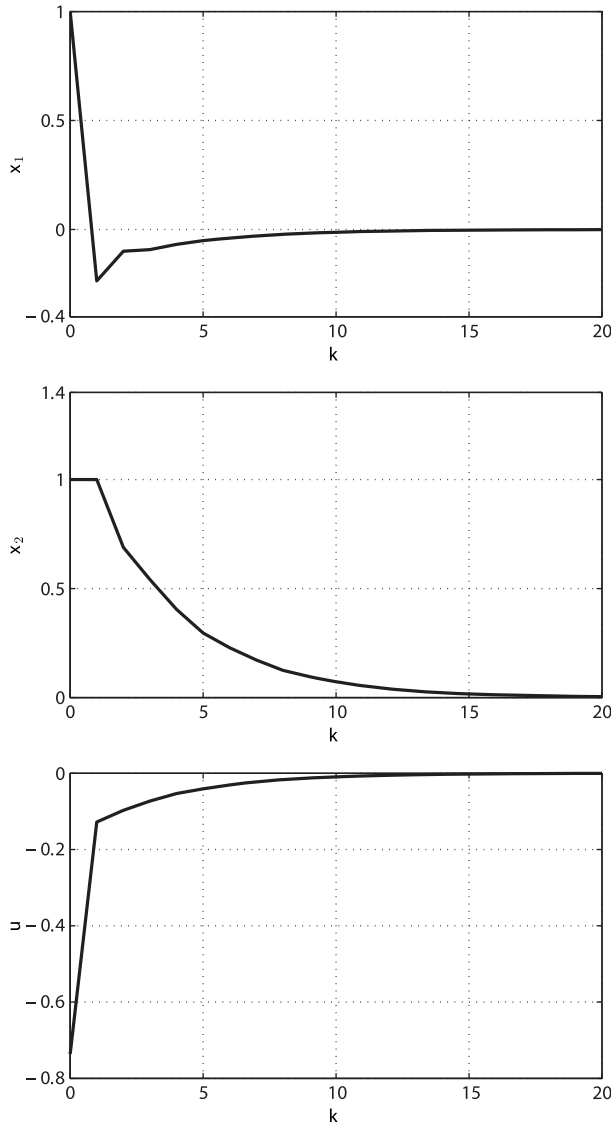
Figure 4. Exemplary time profiles for the closed-loop system from $x_0 = [1\ 1]^T$ related to Problem 4 for the control problem in Example 2.

system into the prescribed polytope (Boyd et al. 1994)

$$
\text{Co}\left\{
\begin{bmatrix} 1/2 & 0 & 1 \\ 1/2 & 1 & 1 \end{bmatrix},
\begin{bmatrix} 1/2 & 0 & 1 \\ 5/6 & 1 & 1 \end{bmatrix},
\begin{bmatrix} -1/2 & 0 & 1 \\ 1/2 & 1 & 1 \end{bmatrix},
\begin{bmatrix} -1/2 & 0 & 1 \\ 5/6 & 1 & 1 \end{bmatrix}
\right\}.
$$

Thus, we can use the proposed MPC scheme according to Algorithm 2. We choose the weighting matrices as $Q = I \in \mathbb{R}^{2 \times 2}$ and $R = 1$, respectively. As discussed in Remark 4.3, Problem 4 is a convex optimisation problem since the input and state constraint sets are convex. To solve the problem, we used CVX, a package for specifying and solving convex

optimisation problem (Grant and Boyd 2008, 2009). Figure 4 shows the state and input trajectory for the system starting from the initial state $x_0 = [1\ 1]^T$, and the optimal performance of Problem 4. For the example system, it is observed that the proposed method achieves good performance as well as constraints satisfaction.

We emphasise that there is *no* feasible solution to the example if the algorithms proposed in Kothare et al. (1996) and Lu and Arkun (2000) are used, where only symmetric box constraints are considered. If we use these approaches to deal with asymmetric constraints, tighter constraints have to be chosen which leads to a smaller feasible set.

6. Summary

In this article, we considered MPC of LPV systems where the time-varying parameter can be measured in real-time and exploited for feedback. We emphasise here that LPV systems are a class of uncertain and time-varying linear systems since the system dynamic at the current time is known and the future system dynamics vary in a pre-specified set. We first presented a closed-loop MPC scheme of constrained LPV systems with symmetric box constraints. The proposed method is based on a parameter-dependent control law which is obtained via the repeated solution of a semi-definite program with respect to LIMs. Closed-loop stability is guaranteed by the feasibility of the LMIs at the initial time instant. Compared with existing algorithms with a static linear control law, the proposed scheme reduces conservativeness and improves performance.

We further proposed an MPC scheme of LPV systems subject to possibly asymmetric constraints, which adopted the analogous framework of a terminal control law, a terminal set and a terminal penalty function of quasi-infinite horizon nonlinear MPC. The optimisation problem was formulated as a convex optimisation problem. It has been shown that recursive feasibility and closed-loop stability are guaranteed by the feasibility of the convex optimisation problem at the initial time instant. For LPV systems with symmetric box constraints, we reduced the convex optimisation problem to a semi-definite program.

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Appendix

Proof of Theorem 1: The proof is divided into three parts in order to show separately that the properties (1)–(3) hold.

(1) Multiplying (10c) from left and right with $\text{diag}\{X_k^{-1}, I, I, I\}$ and substituting $P_k = \gamma_k X_k^{-1}$, $K_{j,k} = Y_{j,k} X_k^{-1}$, we obtain that

$$
\begin{bmatrix}
\gamma_k^{-1} P_k & * & * & * \\
A_{cl}(\theta_{k+v|k}) & \gamma_k P_k^{-1} & * & * \\
Q^{\frac{1}{2}} & 0 & \gamma_k I & * \\
R^{\frac{1}{2}} K(\theta_{k+v|k}) & 0 & 0 & \gamma_k I
\end{bmatrix} \geq 0
$$

holds for all $\theta_{k+v|k} \in \mathcal{P}$, $v \in \mathbb{Z}_{[0,\infty)}$. By the Schur complement, this is equivalent to

$$
A_{cl}^T(\theta_{k+v|k}) P_k A_{cl}(\theta_{k+v|k}) - P_k + Q + K(\theta_{k+v|k})^T R K(\theta_{k+v|k}) \leq 0.
$$

(25)

Multiplying (25) from both sides with $x_{k+v|k}^T$ and $x_{k+v|k}$, respectively, plugging into the system dynamics (1) and using (11), it follows that the inequality

$$x_{k+v+1|k}^T P_k x_{k+v+1|k} - x_{k+v|k}^T P_k x_{k+v|k}$$
$$+ x_{k+v|k}^T Q x_{k+v|k} + u_{k+v|k}^T R u_{k+v|k} \leq 0 \qquad (26)$$

is satisfied for all $v \in \mathbb{Z}_{[0,\infty)}$. Since $Q > 0$ and $R > 0$, clearly $V_{k+v|k} = x_{k+v|k}^T P_k x_{k+v|k}$ is a Lyapunov function and therefore the predicted states $x_{k+v|k}$ converge to zero as $v \to \infty$.

(2) Since $\lim_{v \to \infty} x_{k+v|k} = 0$, by summing up (26) from $v = 0$ to $v = \infty$, we obtain

$$x_{k|k}^T P_k x_{k|k} \geq \sum_{v=0}^{\infty} \left\{ x_{k+v|k}^T Q x_{k+v|k} + u_{k+v|k}^T R u_{k+v|k} \right\}. \qquad (27)$$

Since this inequality is satisfied for all $\theta_{k+v|k} \in \mathcal{P}$, $v \in \mathbb{Z}_{[0,\infty)}$, with $x_{k|k} = x_k$ it follows that

$$V_k = x_k^T P_k x_k \geq J_{\infty|k}(x_k).$$

Thus, V_k is an upper bound on the cost functional (6) at the sampling instant k. Applying the Schur complement to (10b) and substituting $P_k = \gamma_k X_k^{-1}$, we conclude that

$$x_k^T P_k x_k = V_k \leq \gamma_k \qquad (28)$$

holds. Thus, minimising γ_k implies the minimisation of V_k (see also Kothare et al. (1996) for details).

(3) The predicted states and inputs satisfy constraints (3) if

$$x_{k+v|k}^T C_{cl}^T(\theta_{k+v|k}) e_m e_m^T C_{cl}(\theta_{k+v|k}) x_{k+v|k} \leq y_{m,\max}^2, \quad m \in \mathbb{Z}_{[1,n_y]}, \qquad (29)$$

holds for all $\theta_{k+v|k} \in \mathcal{P}$ and all $v \in \mathbb{Z}_{[0,\infty)}$. It follows from (26) and (28) that

$$x_{k+v|k}^T P_k x_{k+v|k} \leq \gamma_k, \quad \forall v \in \mathbb{Z}_{[0,\infty)}. \qquad (30)$$

Thus, inequality (29) is satisfied if

$$\frac{x_{k+v|k}^T C_{cl}^T(\theta_{k+v|k}) e_m e_m^T C_{cl}(\theta_{k+v|k}) x_{k+v|k}}{y_{m,\max}^2} - \frac{x_{k+v|k}^T P_k x_{k+v|k}}{\gamma_k} \leq 0 \qquad (31)$$

holds, which is clearly the case if

$$\frac{P_k}{\gamma_k} - \frac{C_{cl}^T(\theta_{k+v|k}) e_m e_k^T C_{cl}(\theta_{k+v|k})}{y_{m,\max}^2} \geq 0, \quad m \in \mathbb{Z}_{[1,n_z]}, \qquad (32)$$

holds for all $\theta_{k+v|k} \in \mathcal{P}$ and all $v \in \mathbb{Z}_{[0,\infty)}$. Using the definition of $C_{cl}(\theta_{k+v|k})$, with standard modifications we obtain (10d). Thus, satisfaction of the matrix inequalities (10d) implies that (29) holds, and therefore, the constraints (3) are satisfied for all $v \in \mathbb{Z}_{[0,\infty)}$.

Proof of Theorem 2: The proof is divided into four parts in order to show separately that the properties (1)–(4) hold.

(1) It is trivial to show that Problem 3 is convex since the conditions (15b)–(15h) are LMI conditions. By applying Lemma 1 to the LMIs (15c)–(15h) it can be shown that the solution to Problem 3 at the sampling instant k is a feasible solution to Problem 2. Thus, it follows from (26) that

$$x_{k+1|k}^T P_k x_{k+1|k} < x_{k|k}^T P_k x_{k|k} \qquad (33)$$

is satisfied for all k. The first part of the input sequence $u_{k+v|k} = K(\theta_{k+v|k}) x_{k+v|k}$, $v \in \mathbb{Z}_{[0,\infty)}$, obtained at the sampling instant k is applied to the system, i.e. $u_k = K(\theta_k) x_k = K(\theta_{k|k}) x_{k|k} = u_{k|k}$. Furthermore, no model plant mismatch is present, i.e. $x_{k+1|k} = x_{k+1}$. Thus, it follows from (33) that

$$x_{k+1}^T P_k x_{k+1} < x_k^T P_k x_k \qquad (34)$$

holds for all k. This implies that the solution to Problem 3 at the sampling instant k satisfies the LMIs (15b)–(15h) at the sampling instant $k+1$, and therefore is a feasible solution to Problem 3 at the sampling instant $k+1$. It follows by induction that initial feasibility implies feasibility at all future sampling instants.

(2) This property follows directly from the proof of Theorem 1.

(3) It follows that the feedback law $K(\theta_k)$ and the matrix P_k can be calculated at each sampling instant k if Problem 3 is feasible at the first sampling instant. Thus, the expression $V_{k+1} = x_{k+1}^T P_{k+1} x_{k+1}$ is minimised at the sampling instant $k+1$. Since P_k is a feasible, however suboptimal solution to Problem 3 at $k+1$, with (34) it follows that

$$x_{k+1}^T P_{k+1} x_{k+1} \leq x_{k+1}^T P_k x_{k+1} < x_k^T P_k x_k$$

holds for all k. Clearly, $V_k = x_k^T P_k x_k$ is a candidate Lyapunov function and continuous at the origin. Thus, the system (1) is asymptotically stabilised by the control law (16) (Mayne et al. 2000; Rawlings and Mayne 2009).

(4) It follows from the proof of Theorem 1 that at each sampling instant k the predicted state and input trajectories $x_{k+v|k}$ and $u_{k+v|k}$ satisfy constraints (3) for all $v \in \mathbb{Z}_{[0,\infty)}$. Since $u_k = u_{k|k}$ and $x_{k+1|k} = x_{k+1}$, this clearly implies satisfaction of constraints (3) for all k.

Proof of Lemma 2: Summing up the inequality (18) from $i = 1$ to ∞, yields

$$E(x_{\infty/k}) - E(x_{k+1/k})$$
$$\leq - \max_{\theta_{k+i} \in \mathcal{P}} \sum_{i=1}^{\infty} \left\{ x_{k+i/k}^T Q x_{k+i/k} + \pi(x_{k+i/k})^T R \pi(x_{k+i/k}) \right\}. \qquad (35)$$

Since $Q > 0$ and $R > 0$, it follows from (18) that

$$E(x_{k+i+1/k}) - E(x_{k+i/k}) \leq -\lambda_{\min}(Q) \|x_{k+i/k}\|^2, \quad \forall i \in \mathbb{Z}_{[0,\infty)}, \qquad (36)$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of the positive matrix Q. Thus, $E(\cdot)$ is monotonically decreasing.

Since $E(x) \geq 0$ for any $x \in \mathcal{X}_f$ and $E(\cdot)$ is monotonically decreasing, there exists a scalar $C \geq 0$ such that $\lim_{i \to \infty} E(x_{k+i/k}) = C$. Then, for any $\epsilon > 0$, there exists a $N \in \mathbb{Z}_{[0,\infty)}$ such that for all $i \geq N$, $|E(x_{k+i/k}) - C| \leq \frac{\epsilon}{2}$. Consider any adjacent states in the sequence while $i \geq N$,

$$|E(x_{k+i/k}) - E(x_{k+i+1/k})| = |E(x_{k+i/k}) - C - E(x_{k+i+1/k}) + C|$$
$$\leq |E(x_{k+i/k}) - C| + |E(x_{k+i+1/k}) - C|$$
$$= \epsilon.$$

Together with (36), we have

$$\lambda_{\min}(Q) \|x_{k+i/k}\|^2 \leq E(x_{k+i/k}) - E(x_{k+i+1/k}) \leq \epsilon, \quad \forall i \in \mathbb{Z}_{[N,\infty)}.$$

That is, $\lim_{i \to \infty} x_{k+i/k} = 0$. Since $E(x) = x^T P x$, $\lim_{i \to \infty} E(x_{k+i/k}) = 0$. Plugging this into (35) proves the lemma.

Proof of Theorem 3: In what follows, u_k^* and $J_k^0(x_{k/k})$ denote the optimal solution and the optimal cost functional of Problem 4 solved at the sampling time instant k. $x_{k+1/k}^*$ denotes the optimal predicted evolution of the system at the time instant k, that is, $x_{k+1/k}^* = A(\theta_k)x_{k/k} + B(\theta_k)u_{k/k}^*$.

(1) By assumption, there are no disturbances and model mismatch, therefore the state measurement at time $k+1$ is $x_{k+1} = x_{k+1/k}^*$. In virtue of Problem 4, $x_{k+1/k}^* \in \mathcal{X}_f$, i.e. $x_{k+1} \in \mathcal{X}_f$. It follows from Definition 1 that $u_{k+1} = \pi(x_{k+1})$ is a feasible solution to Problem 4 at time instant $k+1$ for all $x_{k+1} \in \mathcal{X}_f$.

(2) Considering the feasible solution obtained at the time instant $k+1$, we have

$$
\begin{aligned}
&J_{k+1}(x_{k+1/k+1}) \\
&= x_{k+1/k+1}^T Q x_{k+1/k+1} + u_{k+1/k+1}^T R u_{k+1/k+1} \\
&\quad + x_{k+2/k+1}^T P x_{k+2/k+1} \\
&= J_k^0(x_{k/k}) - x_{k/k}^T Q x_{k/k} - u_{k/k}^T R u_{k/k} + x_{k+2/k+1}^T P x_{k+2/k+1} \\
&\quad - x_{k+1}^T P x_{k+1} + x_{k+1}^T Q x_{k+1} + \pi^T(x_{k+1}) R \pi(x_{k+1}),
\end{aligned}
$$

with $x_{k+1/k} = x_{k+1/k+1} = x_{k+1}$ and $u_{k+1/k+1} = \pi(x_{k+1})$. In virtue of (18), this implies that

$$
J_{k+1}^0(x_{k+1}) \leq J_{k+1}(x_{k+1/k+1}) \leq J_k^0(x_{k/k}) - x_k^T Q x_k - u_k^{*T} R u_k^*.
\tag{37}
$$

Due to $Q > 0$ and $R > 0$, and the nonincreasing evolution of the optimal cost functional, we infer that $\lim_{k \to \infty} x(k) = 0$. Furthermore, $x = 0$ is asymptotically stable due to the continuity of the optimal value functional J_k^0 in x at 0 (see also Mayne et al. (2000) and Rawlings and Mayne (2009)).

Simple proof of Lemma 4: *The inequalities* (15c)–(15h) *guarantee that the LPV system* (1) *satisfies inequality* (18) *and constraints* (3), *respectively.*

Simple proof of Theorem 6: *Let* $\{\alpha^*, u_k^*, X^*, Y_1^*, Y_2^*, \ldots, Y_N^*\}$ *denote the solution of Problem 5, associated with the minimum cost* $\alpha*$, *at the time instant* k. *We know from Theorem 3 that* $\{\alpha^*, K_k^* x_{k+1/k}, X^*, Y_1^*, Y_2^*, \ldots, Y_N^*\}$ *is a feasible solution for the state* x_{k+1} *at the time instant* $k+1$, *where* $K_k^* = \Sigma_{i=1}^N \theta_{i,k} Y_i^*(X^*)^{-1}$. *The proof of asymptotic stability and constraints satisfaction can be completed along the lines of the proof of Theorem* 3.